Curve Modeling
NURBS

Dr. S.M. Malaek
Assistant: M. Younesi
Motivation
We need more control over the curve
Motivation

Design Sketch
Motivation

Rendered
Motivation
Final Physical Product
Motivation

- Different weights of control points
Motivation

- It is hard to produce exact circle with B-spline.
- Circles can only be represented with rational function (i.e., functions that are quotients of two polynomials).

Four closed B-spline curves with 8 control points:
Motivation

- Circles, ellipses and many other curves that cannot be represented by polynomials.

- we need an extension to B-spline curves.
Motivation

- Generalize B-splines to rational curves using *homogeneous coordinates*:

Non-Uniform Rational B-Splines (NURBS)
NURBS

- Different **weights** of $P_i$, to have different attraction factors.

\[
\begin{align*}
W_1 &= 0.4 \\
W_i &= 1 \\
W_3 &= 3.0
\end{align*}
\]
NURBS

- Different weights of $P_i$, to have different attraction factors.

$W_i=1$  \hspace{1cm} W_0=W_2=W_3=W_5=1, \ W_1=W_4=4$
B-spline Curve: Given \( n+1 \) control points \( P_0, P_1, \ldots, P_n \) and knot vector \( U = \{ u_0, u_1, \ldots, u_m \} \) of \( m+1 \) knots, the B-spline curve of degree \( p \): 

\[
C(u) = \sum_{i=0}^{n} N_{i,p}(u) P_i
\]

Multiply the coordinates of \( P_i \) with a weight \( w \): 

\[
P^w_i = \begin{bmatrix} w_i x_i \\ w_i y_i \\ w_i z_i \\ w_i \end{bmatrix}
\]

B-Spline Curve in homogeneous coordinate
NURBS

\[ C^w(u) = \sum_{i=0}^{n} N_{i,p}(u) w_i \begin{bmatrix} x_i \\ y_i \\ z_i \\ w_i \end{bmatrix} \]

\[ C^w(u) = \begin{bmatrix} \sum_{i=0}^{n} N_{i,p}(u)(w_i x_i) \\ \sum_{i=0}^{n} N_{i,p}(u)(w_i y_i) \\ \sum_{i=0}^{n} N_{i,p}(u)(w_i z_i) \\ \sum_{i=0}^{n} N_{i,p}(u)w_i \end{bmatrix} \]
NURBS

**Convert it back to Cartesian coordinate by dividing $C^w(u)$ with the fourth coordinate:**

$$\mathbf{C}(u) = \sum_{i=0}^{n} \frac{N_{i,p}(u)w_i}{\sum_{j=0}^{n} N_{j,p}(u)w_j} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

$$\mathbf{C}(u) = \frac{1}{\sum_{i=0}^{n} N_{i,p}(u)w_i} \sum_{i=0}^{n} N_{i,p}(u)w_i \mathbf{P}_i$$

$$R_{i,p}(u) = \frac{N_{i,p}(u)w_i}{\sum_{j=0}^{n} N_{j,p}(u)w_j}$$

$$\mathbf{C}(u) = \sum_{i=0}^{n} R_{i,p}(u) \mathbf{P}_i$$
Two Immediate Results
Two Immediate Results

1. If all weights are equal to 1, a NURBS curve reduces to a B-spline curve.

2. NURBS Curves are Rational.

\[
C(u) = \sum_{i=0}^{n} R_{i,p}(u)P_i
\]

\[
R_{i,p}(u) = \frac{N_{i,p}(u)w_i}{\sum_{j=0}^{n} N_{j,p}(u)w_j}
\]
Properties of NURBS Basis Function
Properties of NURBS Basis Function

1. $R_{i,p}(u)$ is a degree $p$ rational function in $u$

2. **Nonnegativity** -- For all $i$ and $p$, $R_{i,p}(u)$ is nonnegative

3. **Local Support** -- $R_{i,p}(u)$ is a non-zero on $[u_i, u_{i+p+1})$

4. On any knot span $[u_i, u_{i+1})$ at most $p+1$ degree $p$ basis functions are non-zero.

5. **Partition of Unity** -- The sum of all non-zero degree $p$ basis functions on span $[u_i, u_{i+1})$ is 1:
Properties of NURBS Basis Function

6. If the number of knots is \( m+1 \), the degree of the basis functions is \( p \), and the number of degree \( p \) basis functions is \( n+1 \), then \( m = n + p + 1 \):

7. Basis function \( R_{i,p}(u) \) is a composite curve of degree \( p \) rational functions with joining points at knots in \([u_i, u_{i+p+1})\).

8. At a knot of multiplicity \( k \), basis function \( R_{i,p}(u) \) is \( C^{p-k} \) continuous.

9. If \( w_i = c \) for all \( i \), where \( c \) is a non-zero constant, \( R_{i,p}(u) = N_{i,p}(u) \).
Properties of \textit{NURBS} Curves
Properties of NURBS Curves

1. **NURBS** curve $C(u)$ is a piecewise curve with each component a degree $p$ rational curve.

2. Equality $m = n + p + 1$ must be satisfied.

3. A clamped **NURBS** curve $C(u)$ passes through the two end control points $P_0$ and $P_n$.

4. **Strong Convex Hull Property**

5. **Local Modification Scheme**

6. $C(u)$ is $C^{p-k}$ continuous at a knot of multiplicity $k$.

7. **B-spline Curves and Bézier Curves Are Special Cases of NURBS Curves**
Properties of NURBS Curves

8. **Modifying Weights**: increasing the value of $w_i$ will pull the curve toward control point $P_i$.

- A NURBS curve of degree 6 and its NURBS basis functions. The selected control point is $P_9$. 
Properties of NURBS Curves

- **Modifying Weights:** decreasing the value of \( w_i \) will push the curve away from control point \( P_i \).

If a weight becomes zero, the coefficient of \( P_i \) is zero and, control point \( P_i \) has no impact on the computation of \( C(u) \) for any \( u \) (i.e., \( P_i \) is "disabled").
Properties of NURBS Curves

<table>
<thead>
<tr>
<th>(u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6)</th>
<th>(u_7)</th>
<th>(u_8)</th>
<th>(u_9 = u_{10} = u_{11} = u_{12} = u_{13} = u_{14} = u_{15})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/3</td>
<td>2/3</td>
<td>1</td>
</tr>
</tbody>
</table>

A NURBS curve of degree 6 defined by 9 control points (\(n = 8\)) and 16 knots (\(m = 15\)). The selected control point is \(P_4\). Since the coefficient of \(P_4\), \(N_{4,6}(u)\), is non-zero on \([u_4, u_{4+6+1}] = [0,1)\), changing \(w_4\) affects the entire curve!
Rational Bézier Curves
Rational Bézier Curves

- We have learned that projecting a 4-dimensional B-spline curve to hyperplane $w=1$ yields a 3-dimensional NURBS curve.

What if this B-spline curve is a Bézier curve?

The result is a Rational Bézier curve!
Rational Bézier Curves

Since a rational Bézier curve is a special case of NURBS curves, rational Bézier curves satisfy all important properties that NURBS curves have.

- since there is no internal knots, rational Bézier curves do not have the local modification property, which means modifying a control point or its weight will cause a global change

\[
C(u) = \sum_{i=0}^{n} R_{i,n}(u) P_i
\]

\[
R_{i,n}(u) = \frac{B_{i,n}(u)w_i}{\sum_{j=0}^{n} B_{n,j}(u)w_j}
\]
Rational Bézier Curves

\[
C(u) = \sum_{i=0}^{n} R_{i,n}(u)P_i
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\[
R_{i,n}(u) = \frac{B_{i,n}(u)w_i}{\sum_{j=0}^{n} B_{n,j}(u)w_j}
\]

Modifying the weight of a control point will push or pull the curve away from or toward the control point.
Rational Bézier Curves: Conic Sections
Rational Bézier Curves: Conic Sections

we use rational Bézier curve of degree 2, the coefficients are:

\[ B_{2,0}(u) = (1 - u)^2 \]
\[ B_{2,1}(u) = 2(1 - u)u \]
\[ B_{2,2}(u) = u^2 \]

\[ R_{i,n}(u) = \frac{B_{i,n}(u)w_i}{\sum_{j=0}^{n} B_{n,j}(u)w_j} \]

\[ C(u) = \sum_{i=0}^{n} R_{i,n}(u)P_i \]
Rational Bézier Curves: Conic Sections

The equation of this rational Bézier curve of degree 2 is:

\[
C(u) = \frac{1}{(1-u)^2 + 2(1-u)uw + u^2} \left( (1-u)^2 P_0 + 2(1-u)uwP_1 + u^2 P_2 \right)
\]

\[
w = \frac{r}{1-r}, \quad 0 \leq r < 1
\]

<table>
<thead>
<tr>
<th>Conic Section</th>
<th>( w )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbola</td>
<td>( w &gt; 1 )</td>
<td>( r &gt; 1/2 )</td>
</tr>
<tr>
<td>Parabola</td>
<td>( w = 1 )</td>
<td>( r = 1/2 )</td>
</tr>
<tr>
<td>Ellipse</td>
<td>( w &lt; 1 )</td>
<td>( r &lt; 1/2 )</td>
</tr>
<tr>
<td>Straight line</td>
<td>( w = 0 )</td>
<td>( r = 0 )</td>
</tr>
</tbody>
</table>

\( P_0 = 1 \)
\( P_1 = w \)
\( P_2 = 1 \)
### Rational Bézier Curves: Conic Sections

<table>
<thead>
<tr>
<th>Type</th>
<th>Weight Condition</th>
<th>Parameter Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hyperbola</td>
<td>$W &gt; 1$</td>
<td>$r &gt; 1/2$</td>
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</table>

- $P_0 = 1$
- $P_1 = w$
- $P_2 = 1$
Weights $w$ is equal to the sine of the half angle at control point $P_1$:

$$w = \sin(a)$$
Rational Bézier Curves: Conic Sections

Quarter Circle:
- $P_1 = 90^\circ$, $a = 45^\circ$,

$$w = \sin(45^\circ) = \frac{\sqrt{2}}{2}$$
Rational Bézier Curves: Conic Sections

1/3 Circle:
- $P_1 = 60^\circ$, $a = 30^\circ$,

$$w = \sin(30^\circ) = 1/2$$
Rational Bézier Curves: Conic Sections

Full Circle:

\[
U = \begin{bmatrix}
0, 0, 0, & \frac{1}{3}, & \frac{1}{3}, & \frac{2}{3}, & \frac{2}{3}, & 1, 1, 1
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
0, 0, 0, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{3}{4}, & \frac{3}{4}, & 1, 1, 1
\end{bmatrix}
\]